Green's Function and Poisson Kernel on a Path

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2000 Mathematics Subject Classification. 39A

Green's function and Poisson kernel of networks can be used to study diffusion problems on graphs, such as hitting time, chip-firing and discrete Markov chains. The reason is that these problems are written in terms of boundary value problems with respect to the Laplace operator associated with the network and the Green's function or the Poisson kernel are the resolvent kernels. Therefore, Green's functions and Poisson kernels constitute a powerful tool in dealing with a wide range of combinatorial problems, see [2, 3].

The authors studied in [2] general boundary value problems on arbitrary subsets of a network and in particular they obtained a relation between the Green's function and the Poisson kernel. In general it is not easy to obtain and explicit expression for that functions. However, when the network presents symmetries they can be obtained by hand.

Our objective here is to obtain explicit expressions for the Green function and Poisson kernel for the Sturm-Liouville problem associated with the Schrödinger operator on the path $P_n = \{0, \ldots, n+1\}$. In this case the explicit construction of such functions is a mechanic process since one has two independent solutions of the homogeneous problem associated with the differences equation. Specifically, they are given through Chebyshev Polynomials, since they verified a recurrence relation of the type $P_{i+2}(x) = 2\,x\,P_{i+1}(x) - P_i(x)$, $i \in Z$.

For instance, if we consider the Neumann regular problem, that is,

$$2(1+q)z(i)-z(i+1)-z(i) = f(i)\ ,i = 1, \ldots, n, \qquad z(0)-z(1) = z(n+1)-z(n) = 0,$$

where $q > 0$, then the Green function is given by

$$G_q(k,s) = \frac{1}{2q\,U_{n-1}(1+q)} \begin{cases} V_{k-1}(1+q)\,V_{n-s}(1+q)\,, & 0 \le k \le s \le n, \\ V_{s-1}(1+q)\,V_{n-k}(1+q)\,, & 1 \le s \le k \le n+1, \end{cases}$$

where U_k and V_k denote the Second and Third kind Chebyshev polynomials, respectively, [1].

[1] Mason, J.C. and Handscomb, D.C., *Chebyshev Polynomials*. Chapman & Hall, CRC Press Company, 2003.

[2] Bendito, E., Carmona, A., Encinas, A.M., Solving boundary value problems on networks using equilibrium measures, *J. Funct. Anal.* **17** (2000), 155–176.

[3] Chung, F.R.K., Yau, S.T., Discrete Green functions, *J. Comb. Theory* **91** (2000), 191–214.